# ON THE THEORY OF A PENDULUS GYROSCOPIC SYSTEM, SATISFYING THE CONDITIONS OF M.SCHULER

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We investigate here a mechanical system with gyroscopes which is on the three degrees of freedom suspension and moves near the Earth's surface. The center of mass of the system and the center of suspension do not coincide. A special case of such a mechanical system is a system of a horizontal gyrocompass or a two-gyroscope vertical. The theory of these systems has been given by Ishlinskii in 1 and 2 . Klimov in 3 obtained the conditions which must satisfy an arbitrary moving pendulous gyroscopic system in order to have a position of relative equilibrium at which the z-axis of the system, passing through the center of mass and the center of suspension, coincides with the line directed toward the Earth's center.

We shall demonstrate that the conditions of Klimov remain valid for nonsphericity of the Earth gravitational field, and investigate the perturbed motion of the pendulous gyroscopic system, meaning its oscillations about its position of relative equilibrium. We shall demonstrate that this motion can be reduced to the motion of the physical pendulum investigated in 4 and 5 .

1. Let us consider a three degrees of freedom mechanical system placed on a moving object in such a way that the center of mass of the system and the center of suspension do not coincide. The system may contain different mechanical gadgets which move with respect to each other, among others, gyroscopes. Such a mechanical system can be called a pendulous gyroscopic system.

Let us introduce in our system the trihedron $Oxyz$ with its origin in the center of suspension, the z-axis coinciding with the line from the center of mass to the center of suspension. Let $a$ be the distance between the center of mass $c$ and the center of suspension $O$ . Then the coordinates of the center of mass are

$$x_c = y_c = 0, \qquad z_c = -a \tag{1.1}$$

We shall find conditions under which the z-axis of the system in the position of relative equilibrium coincides with the line in the direction toward the Earth's center.

Using the angular momentum theorem for the whole system as it moves about the center of mass we obtain

$$\mathbf{K}^{\cdot} = \mathbf{M} \tag{1.2}$$

Here and further on the dot following a letter will signify a time derivative. $K$ is the total angular momentum of the system, $M$ is the total moment with respect to the center of suspension. The total moment $M$ is created by the Earth's attraction forces and the inertia forces of the translatory motion of the center of suspension.

We shall reduce the action of the attraction forces to the one force F applied at the center of mass and coinciding with the direction of the gravitational field at the point O . The nonhomogeneity of the gravitational field inside the body is neglected. We shall add to the moment of the force F the moment of the inertia forces of the translatory motion and also an artificially constructed moment $M^*$ . From (1.1) and (1.2) we obtain

$$\mathbf{K}^{\cdot} = \mathbf{a} \times (\mathbf{F} - m\mathbf{r}^{\cdot\cdot}) + \mathbf{M}^* \tag{1.3}$$

Here r is the radius vector of the point O originating at the Earth's center $O_1$ , a is the radius vector at the point C originating at the point O . Let the trihedron xyz in the position of relative equilibrium be denoted by $x_0y_0z_0$ . Then

$$\mathbf{F} = F_{x_0}\mathbf{x}_0 + F_{y_0}\mathbf{y}_0 + F_{z_0}\mathbf{z}_0, \qquad \mathbf{M}^* = M_{x_0}{}^*\mathbf{x}_0 + M_{y_0}{}^*\mathbf{y}_0 + M_{z_0}{}^*\mathbf{z}_0 \tag{1.4}$$

Here $x_0y_0z_0$ are the unit vectors along the corresponding axes. Since in the position of relative equilibrium the vector a is collinear with vector r , it follows by (1.3) and (1.4) that

$$\mathbf{K}^{\cdot} = -\mathbf{a} \times m\mathbf{r}^{\cdot\cdot} + (M_{x_0}{}^* - aF_{y_0})\,\mathbf{x}_0 + (M_{y_0}{}^* + aF_{x_0})\,\mathbf{y}_0 + M_{z_0}{}^*\mathbf{z}_0 \tag{1.5}$$

Let

$$M_{y_0}{}^* = -aF_{x_0}, \qquad M_{x_0}{}^* = aF_{y_0}, \qquad M_{z_0}{}^* = 0 \tag{1.6}$$

Then Equation (1.5) becomes

$$\mathbf{K}^{\cdot} = -\mathbf{a} \times m\mathbf{r}^{\cdot\cdot} = 0 \tag{1.7}$$

We require that the condition

$$a = kr \tag{1.8}$$

be satisfied, that is we require that the distance between the center of mass and the center of suspension be proportional to the distance to the Earth's center. In order to satisfy this condition when r is variable we must have an input with the values of r .

If (1.8) is satisfied, then in the position of relative equilibrium $\mathbf{a} = -k\mathbf{r}$, $\mathbf{a}^{\cdot} \times \mathbf{r}^{\cdot} = 0$, hence Equation (1.7) can be written in the form

$$\mathbf{K}^{\cdot} = km\,(\mathbf{r} \times \mathbf{r}^{\cdot})^{\cdot} \tag{1.9}$$

Now it can be integrated from which we obtain

$$\mathbf{K} - m\mathbf{a} \times \mathbf{r}^{\cdot} = \mathbf{b} = \text{const} \tag{1.10}$$

Substituting b = 0 , introducing the notation $\mathbf{v} = \mathbf{r}^{\cdot}$ and projecting on the axes $x_0y_0z_0$ , we find

$$K_{x_0} + mav_{y_0} = 0, \qquad K_{y_0} - mav_{x_0} = 0 \qquad K_{z_0} = 0 \tag{1.11}$$

The conditions (1.6), (1.8), (1.11) appear to be the conditions which the parameters of the system must satisfy, to make the z-axis coincide with the line in the direction toward the Earth's center in the position of relative equilibrium. Besides, the initial conditions must also be satisfied. Namely, at the initial moment the vector $a$ should be oriented along $r$ , and the initial rate of change of the orientation of the vector a should equal the initial rate of change of the orientation of the vector r .

The conditions (1.8) and (1.11) were derived by Klimov 3 who used a different notation from ours. We repeated here the derivation of (1.8) and (1.11) because later on we need certain intermediate relations arising in the derivation. We shall also show that nonsphericity of the Earth gravitational field does not change the conditions (1.8), (1.11), and requires only the supplementary conditions (1.6).

From the conditions (1.6), (1.8), (1.11) we obtain as special cases the conditions for the existence of the position of relative equilibrium of a physical pendulum 4 and 5, of a horizontal gyrocompass 1, and a two-gyroscopic vertical 2.

2. We shall derive the equations for small oscillations of our pendulous gyroscopic system about its position of relative equilibrium. These oscillations occur when the conditions (1.6), (1.8), (1.11) and initial conditions are not exactly satisfied. Varying Equation (1.2) in the neighborhood of the relative equilibrium we have

$$\delta\mathbf{K}^{\cdot} = \delta\mathbf{M} \tag{2.1}$$

The angular momentum K is derived from (1.10) where $\mathbf{b} = 0$, $\mathbf{a} = -k\mathbf{r}$. Thus

$$\delta\mathbf{K} = mk\delta\mathbf{r} \times \mathbf{r}^{\cdot} + mk\mathbf{r} \times \delta\mathbf{r}^{\cdot} + \Delta\mathbf{K} \tag{2.2}$$

Here $\Delta\mathbf{K}$ is a certain instrumental error

$$\mathbf{r} = r\mathbf{z}_0, \qquad \delta\mathbf{r} = \delta x\mathbf{x}_0 + \delta y\mathbf{y}_0 + \Delta r\mathbf{z}_0 \tag{2.3}$$

where $\Delta r$ is the error in the input of r.

In order to obtain $\delta\mathbf{M}$, we notice that the variation of the moment is determined from the variation $\delta\mathbf{a}$, the variation of the correcting moment $\delta\mathbf{M}^*$ and from the moment $\Delta\mathbf{M}$ arising from instrumental errors (for example the fractional moment in the suspension or the moment of the residual inbalance). Consequently, taking into account the expression for M contained in the right-hand side of Equation (1.3) we find

$$\delta\mathbf{M} = \delta\mathbf{a} \times (\mathbf{F} - m\mathbf{r}^{\cdot\cdot}) + \delta\mathbf{M}^* + \Delta\mathbf{M} \tag{2.4}$$

where the quantity $\delta\mathbf{a}$ is determined from Equation

$$\delta\mathbf{a} = -k\delta\mathbf{r} - \Delta a\mathbf{z}_0 \tag{2.5}$$

The first term in the right-hand side is caused by the error in the input of r and by the deviation of the z-axis from the direction toward the Earth's center, the second term by the instrumental error $\Delta a$ arising from the inaccurate setting of the distance a between the center of the suspension and the center of mass of the system. Differentiating (2.2) and observing that

$$\delta\mathbf{r}^{\cdot} \times \mathbf{r}^{\cdot} + \mathbf{r}^{\cdot} \times \delta\mathbf{r}^{\cdot} = 0 \tag{2.6}$$

we find

$$\delta\mathbf{K}^{\cdot} = km\delta\mathbf{r} \times \mathbf{r}^{\cdot\cdot} + km\mathbf{r} \times \delta\mathbf{r}^{\cdot\cdot} + \Delta\mathbf{K}^{\cdot} \tag{2.7}$$

We have to substitute now (2.7) and (2.4) in (2.1).

Before performing this substitution it is convenient to simplify a little Expression (2.4), in order to make the variation of the moment easier. Prior to that we can also neglect the small variation 4 of the moment $\mathbf{M}^*$, which corrects the action of the horizontal component of the gravitational field. Returning to the first equation of (1.4) we obtain

$$\delta\mathbf{a} \times \mathbf{F} = \delta\mathbf{a} \times (F_{x_0}\mathbf{x}_0 + F_{y_0}\mathbf{y}_0 + F_{z_0}\mathbf{z}_0) \tag{2.8}$$

Neglecting in the right-hand side the product of $\delta\mathbf{a}$ by the sum of the first and the second term inside parantheses, which is of the second order of smallness, and observing that using the same accuracy we can set $F_{z_0} = -\mu m / r^2$, where m is the product of the gravitational constant by the Earth's mass, we find

$$\delta\mathbf{M} = \delta\mathbf{a} \times (-\mu m r^{-2}\mathbf{z}_0 - m\mathbf{r}^{\cdot\cdot}) + \Delta\mathbf{M} \tag{2.9}$$

Substituting (2.5) into (2.9) we obtain Formula

$$\delta\mathbf{M} = mk\delta\mathbf{r} \times (\mu r^{-2}\mathbf{z}_0 + \mathbf{r}^{\cdot\cdot}) + m\Delta a\mathbf{z}_0 \times \mathbf{r}^{\cdot\cdot} + \Delta\mathbf{M} \tag{2.10}$$

We substitute now (2.10) and (2.7) into (2.1). After collecting similar terms and grouping them in order to be able to use the first Equation (2.3) we obtain Equation

$$km\mathbf{r} \times (\delta\mathbf{r}^{\cdot\cdot} + \mu r^{-3}\delta\mathbf{r}) = -\Delta\mathbf{K}^{\cdot} + \Delta\mathbf{M} + m\Delta a z_0 \times \mathbf{r}^{\cdot\cdot} \tag{2.11}$$

This equation turns out to be the vector equation for small oscillations about the position of relative equilibrium of the pendulous gyroscopic system satisfying (with certain errors) the conditions (1.6), (1.8) and (1.11).

3. In order to obtain the scalar equations we must substitute in (2.11) the expressions for $r$ and $\delta r$ from (2.3) and project the result on the axes $x_0, y_0, z_0$ taking also into account

$$z_0^{\cdot}\cdot x_0 = \omega_{y_0}, \quad z_0^{\cdot\cdot}\cdot x_0 = \omega_{y_0}^{\cdot} + \omega_{x_0}\omega_{z_0}, \quad z_0^{\cdot}\cdot y_0 = -\omega_{x_0}, \quad z_0^{\cdot\cdot}\cdot y_0 = -\omega_{x_0}^{\cdot} + \omega_{y_0}\omega_{z_0} \quad (3.1)$$

where $\omega_{x_0}, \omega_{y_0}, \omega_{z_0}$ are the components of the absolute angular velocity of the trihedron $x_0y_0z_0$ on its axes. These substitutions and projections result, after appropriate grouping and simplifications, in Equations

$$\begin{aligned} \delta x^{\cdot\cdot} + (\mu r^{-3} - \omega_{y_0}^2 - \omega_{z_0}^2)\,\delta x + (\omega_{x_0}\omega_{y_0} - \omega_{z_0}^{\cdot})\,\delta y - 2\omega_{z_0}\delta y^{\cdot} = -\Delta r\,(\omega_{x_0}\omega_{z_0} + \omega_{y_0}^{\cdot}) - \\ - 2\omega_{y_0}\Delta r^{\cdot} + (ma)^{-1}(-\Delta K_{y_0}^{\cdot} - \omega_{z_0}\Delta K_{x_0} + \omega_{x_0}\Delta K_{z_0}) + (ma)^{-1}\Delta M_{y_0} + \\ + a^{-1}\Delta a\,[r\,(\omega_{y_0}^{\cdot} + \omega_{x_0}\omega_{z_0}) + 2r^{\cdot}\omega_{y_0}] \end{aligned} \quad (3.2)$$

$$\begin{aligned} \delta y^{\cdot\cdot} + (\mu r^{-3} - \omega_{x_0}^2 - \omega_{z_0}^2)\,\delta y + (\omega_{x_0}\omega_{y_0} + \omega_{z_0}^{\cdot})\,\delta x + 2\omega_{z_0}\delta x^{\cdot} = -\Delta r\,(\omega_{y_0}\omega_{z_0} - \omega_{x_0}^{\cdot}) + \\ + 2\omega_{x_0}\Delta r^{\cdot} + (ma)^{-1}(\Delta K_{x_0}^{\cdot} + \omega_{y_0}\Delta K_{z_0} - \omega_{z_0}\Delta K_{y_0}) - (ma)^{-1}\Delta M_{x_0} - \\ - a^{-1}\Delta a\,[r\,(\omega_{x_0}^{\cdot} - \omega_{y_0}\omega_{z_0}) + 2r^{\cdot}\omega_{x_0}] \end{aligned}$$

$$\Delta K_{z_0}^{\cdot} + \omega_{x_0}\Delta K_{y_0} - \omega_{y_0}\Delta K_{x_0} = \Delta M_{z_0}$$

Let $a$ and $b$ be the angles of small deviations of the z-axis from the direction toward the Earth's center in the planes $z_0x_0$ and $z_0y_0$; respectively. It is obvious that $\alpha = r^{-1}\delta x$, $\beta = -r^{-1}\delta y$. Consequently, Equations (3.2) control the small oscillations of the z-axis about the line directed toward the Earth's center. The homogeneous system of the first two equations (3.2) is the same as for the small oscillations of the pendulum of Schuler 4 and also the same as the first group of equations for the errors of an inertial system with two accelerometers 6 . The equations for small oscillations in a two-gyroscopic vertical and in a horizontal gyrocompass obtained in 1 and 2 can also be reduced to the system of the first two equations (3.2).

Let us draw attention to the following interesting circumstance. In the papers 1 and 2 the equations for small oscillations were obtained from the precessional formulation of the problem, whereas our Equations (3.2) which control small oscillations of a horizontal gyrocompass and a two-gyroscopic vertical were derived without using the precessional theory. Consequently, the homogeneous equations for small oscillations of the z-axis in a horizontal gyrocompass and in a two-gyroscopic vertical which were obtained in 1 and 2 by using the precessional theory can be obtained and are valid outside the precessional theory if all the three conditions (1.12) are satisfied.

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